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Abstract

Metric inequalities, cutset inequalities and Benders feasibility cuts are three families of valid inequalities that have been widely used in different algorithms for network design problems. This article sheds some light on the interrelations between these three families of inequalities. In particular, we show that cutset inequalities are a subset of the Benders feasibility cuts, and that Benders feasibility cuts (including cutset inequalities) are not, in general, metric inequalities. We also propose a simple procedure to convert Benders feasibility cuts into stronger metric inequalities. Computational results show the efficiency of the strengthening procedure.

Keywords: multicommodity capacitated network design, metric inequalities, cutset inequalities, Benders decomposition.

Résumé

Les inégalités métriques, les inégalités de coupes et les contraintes d'admissibilité de Benders sont trois familles d'inégalités qui ont été largement utilisées dans divers algorithmes pour les problèmes de conception de réseaux. Cet article étudie les liens qui existent entre ces familles d'inégalités. En particulier, nous montrons que les inégalités de coupes sont un sous-ensemble des contraintes de Benders et que les contraintes de Benders (incluant les inégalités de coupes) ne sont pas, en général, des inégalités métriques. Nous proposons aussi une procédure simple afin de convertir les contraintes de Benders en des inégalités métriques plus fortes. Les résultats d'expériences de calcul montrent l'efficacité de cette procédure.

Mots-clés: conception de réseaux multiflots avec capacité, inégalités métriques, inégalités de coupes, décomposition de Benders.

1 Introduction

Network design problems have received significant attention in the literature for more than two decades, as witnessed by the reviews of Magnanti and Wong (1984), Minoux (1989), Balakrishnan, Magnanti and Mirchandani (1997) and Gendron, Crainic and Frangioni (1999). In particular, multicommodity capacitated network design problems have received considerable attention (see, e.g., Stoer and Dahl, 1994; Gabrel, Knippel and Minoux, 1999; Günlük, 1999; Atamtürk, 2002a, 2002b).

In most of these papers, frequent reference is made to several types of valid inequalities which are used to improve LP relaxation bounds, prove theoretical results or accelerate the convergence of solution algorithms. In this article, we study three families of such inequalities: *metric inequalities*, *cutset inequalities* and *Benders feasibility cuts*. Although these inequalities have been studied individually, very few has been said about the interrelations between them, except from the fact that the cutset inequalities are a subset of the metric inequalities (see, e.g., Stoer and Dahl, 1994).

In this article, we show that cutset inequalities are, in fact, a subset of Benders feasibility cuts and, although it is true that some of them are also metric inequalities, this relation is not valid in general, i.e., there exist cutset inequalities that are *not* metric inequalities. Moreover, we show that it is easy to obtain a procedure that will strengthen Benders feasibility cuts in order to obtain the associated metric inequalities. This procedure can also be used to strengthen cutset inequalities, as they are a subset of Benders feasibility cuts.

In this article, all developments are made with respect to the *Multicommodity Capacitated Fixed-charge Network Design Problem* (MCFND), for which several exact and approximate solutions procedures have been proposed (Gendron and Crainic, 1994; Gendron and Crainic, 1996; Holmberg and Yuan, 2000; Crainic, Gendreau and Farvolden, 2000; Sellman, Kliewer and Koberstein, 2002; Ghamlouche, Crainic and Gendreau, 2003; Ghamlouche, Crainic and Gendreau, 2004; Crainic, Gendron and Hernu, 2004). Note that the results are more general and apply to other multicommodity capacitated network design models. The following notation is used: $G(N, A, K)$ is a directed graph where N is the set of nodes, A is the set of arcs, and K is the set of commodities to be routed in the network. Each commodity k can be represented by the triplet $(O(k), D(k), d_k)$, where $O(k)$ is the origin node, $D(k) \neq O(k)$ is the destination node and d_k is the demand. To each arc $(i, j) \in A$ are associated a binary variable y_{ij} taking value 1 if and only if the arc is chosen, and continuous variables x_{ij}^k, $k \in K$, representing the flow of commodity k on the arc. Parameters f_{ij} and c_{ij}^k are the fixed cost of opening arc (i, j) and the variable cost associated with the flow of one unit of commodity k through arc (i, j), respectively, while u_{ij} is the capacity of arc (i, j) when open. With this notation, one can write

the MCFND as follows:

$$\text{Minimize} \sum_{(i,j)\in A} \left(f_{ij} y_{ij} + \sum_{k\in K} c_{ij}^k x_{ij}^k \right) \tag{1}$$

subject to

$$\sum_{j\in N_i^+} x_{ij}^k - \sum_{j\in N_i^-} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \notin \{O(k), D(k)\}, \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N, \quad \forall k \in K, \tag{2}$$

$$\sum_{k\in K} x_{ij}^k \leq u_{ij} y_{ij}, \qquad \forall (i,j) \in A, \tag{3}$$

$$x_{ij}^k \geq 0, \qquad \forall (i,j) \in A, \quad \forall k \in K, \tag{4}$$

$$y_{ij} \in \{0,1\}, \qquad \forall (i,j) \in A, \tag{5}$$

where $N_i^+ = \{j | (i,j) \in A\}$ and $N_i^- = \{j | (j,i) \in A\}$.

The objective function (1) minimizes the sum of fixed costs associated with the opening of the links and variable costs associated with the flow of commodities. Constraints (2) are responsible for the flow conservation at each node and also ensure the satisfaction of the demand requirements. Finally, constraints (3) are the capacity constraints that limit the flow on each arc to the value of its capacity (if the arc is open) or to zero (if the arc is closed).

This paper is organized as follows. In the next section, we introduce the Benders reformulation of (1)-(5) in order to obtain the Benders feasibility cuts. Then, in Section 3, we show that cutset inequalities are a subset of the latter cuts. Section 4 develops a simple procedure to strengthen Benders feasibility cuts and obtain metric inequalities, while Section 5 tests this procedure both for the case of the general Benders feasibility cuts and for the subset of cutset inequalities. Section 6 ends the paper with some conclusions.

2 Benders feasibility cuts

In this section we reformulate model (1)-(5) to obtain a new formulation that is based solely on the y_{ij} variables and a real variable z. The price paid for excluding the flow variables is that the new formulation comprises a large number of constraints. The idea of reformulating mixed-integer linear problems in this fashion was first proposed by Benders (1962).

First, one defines the *primal subproblem*, that is a problem on the continuous flow variables, given any binary vector $\overline{y} = [\overline{y}_{ij}]$:

$$v(\overline{y}) = \text{Minimize} \sum_{(i,j)\in A} \sum_{k\in K} c_{ij}^k x_{ij}^k \tag{6}$$

subject to

$$\sum_{j\in N_i^+} x_{ij}^k - \sum_{j\in N_i^-} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \notin \{O(k), D(k)\},\ \forall i \in N,\ \forall k \in K, \\ -d_k, & i = D(k), \end{cases} \tag{7}$$

$$\sum_{k\in K} x_{ij}^k \leq u_{ij}\overline{y}_{ij}, \qquad \forall (i,j) \in A, \tag{8}$$

$$x_{ij}^k \geq 0, \qquad \forall (i,j) \in A,\ \forall k \in K. \tag{9}$$

When the values of the y_{ij} variables are fixed, constraints (8) become simple upper bounds on the value of the flows, i.e., once a tentative network is given, the primal subproblem minimizes the variable cost by finding values for the flow variables.

Associating dual variables $\pi = [\pi_i^k]$ and $\alpha = [\alpha_{ij}]$ to constraints (7) and (8), respectively, one can write the dual of (6)-(9), named the *dual subproblem*:

$$\text{Maximize} \sum_{k\in K} d_k(\pi_{O(k)}^k - \pi_{D(k)}^k) + \sum_{(i,j)\in A} u_{ij}\overline{y}_{ij}\alpha_{ij} \tag{10}$$

subject to

$$\pi_i^k - \pi_j^k + \alpha_{ij} \leq c_{ij}^k, \qquad \forall (i,j) \in A,\ \forall k \in K, \tag{11}$$

$$\alpha_{ij} \leq 0, \qquad \forall (i,j) \in A. \tag{12}$$

Let D be the feasible region of the dual subproblem defined by (11) and (12), and let P_D and Q_D be the set of extreme points and extreme rays of D, respectively. Since $c_{ij}^k \geq 0$, $D \neq \emptyset$ and thus by strong duality the primal subproblem is either feasible and bounded or infeasible. These two situations can be characterized as follows. If

$$\sum_{k\in K} d_k(\pi_{O(k)}^k - \pi_{D(k)}^k) + \sum_{(i,j)\in A} u_{ij}\overline{y}_{ij}\alpha_{ij} \leq 0 \tag{13}$$

for every extreme ray $(\pi, \alpha) \in Q_D$, then the dual subproblem is bounded and the value of both the dual and the primal subproblems is given by:

$$\max_{(\pi,\alpha)\in P_D} \sum_{k\in K} d_k(\pi_{O(k)}^k - \pi_{D(k)}^k) + \sum_{(i,j)\in A} u_{ij}\overline{y}_{ij}\alpha_{ij}. \tag{14}$$

However, if there exists an extreme ray $(\pi, \alpha) \in Q_D$ such that:

$$\sum_{k\in K} d_k(\pi_{O(k)}^k - \pi_{D(k)}^k) + \sum_{(i,j)\in A} u_{ij}\overline{y}_{ij}\alpha_{ij} > 0, \tag{15}$$

then the dual subproblem is unbounded and the primal subproblem infeasible. Introducing a free variable z, the original problem can thus be rewritten as:

$$\text{Minimize} \sum_{(i,j)\in A} f_{ij}y_{ij} + z \tag{16}$$

subject to

$$\sum_{(i,j)\in A} u_{ij}\alpha_{ij}y_{ij} \leq \sum_{k\in K} d_k(\pi^k_{D(k)} - \pi^k_{O(k)}),\ \forall(\pi,\alpha) \in Q_D \tag{17}$$

$$z - \sum_{(i,j)\in A} u_{ij}\alpha_{ij}y_{ij} \geq \sum_{k\in K} d_k(\pi^k_{O(k)} - \pi^k_{D(k)}),\ \forall(\pi,\alpha) \in P_D \tag{18}$$

$$y_{ij} \in \{0,1\},\ \forall(i,j) \in A. \tag{19}$$

Constraints (17) and (18) are known as Benders *feasibility cuts* and Benders *optimality cuts*, respectively.

3 Benders feasibility cuts and cutset inequalities

Let $S \subset N$ be a non-empty subset of N and $\overline{S} = N \setminus S$ its complement. Also let $K(S,\overline{S}) \subseteq K$ be the set of commodities having their origin in S and their destination in $\overline{S}$. For any set S such that $K(S,\overline{S})$ is non-empty, the following inequality is valid:

$$\sum_{(i,j)\in(S,\overline{S})} u_{ij}y_{ij} \geq d_{(S,\overline{S})}, \tag{20}$$

where $d_{(S,\overline{S})} = \sum_{k\in K(S,\overline{S})} d^k$ and $(S,\overline{S})$ is a cutset, i.e., the set of arcs that connect a node in S to a node in $\overline{S}$. Valid inequalities of form (20) are known as *cutset inequalities*.

We now show that cutset inequalities can be viewed as feasibility constraints (17) obtained when particular values are given to the dual variables π and α.

Proposition 1 *For $S \subset N$, set:*

$$\pi^k_i = \begin{cases} 1, & \text{if } i \in S \text{ and } O(k) \in S \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{ij} = \begin{cases} -1, & \text{if } i \in S \text{ and } j \in \overline{S}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the feasibility cut (17) is equivalent to the cutset inequality (20) for the node subset S.

It is easy to see that the proposition is true by simply comparing the terms of the concerned feasibility constraint (17) and the cutset inequality. Therefore, one can say that the cutset inequalities are a subset of Benders feasibility cuts.

It is worth noting that in the case of the single-commodity problem, the Benders feasibility cuts are equivalent to the cutset inequalities. Indeed, in this case, all extreme rays of the polyhedron defined by (11)-(12) are associated with a cutset inequality, as shown in the following proposition.

Proposition 2 *In the single-commodity capacitated fixed-charge network design problem, the extreme rays* (π, α) *of the dual subproblem (11)-(12) are of the form:*

$$\pi_i = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{ij} = \begin{cases} -1, & \text{if } i \in S \text{ and } j \in \overline{S}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: First we show that the all the positive π variables have the same value. Indeed, suppose that there exists an extreme ray, $r = (\pi_i^r, \alpha_{ij}^r)$, with at least two positive π_i^r variables taking different values. Also let $\underline{\pi}$ be the smallest value among the positive π variables. One can then decompose r in two feasible rays r_1 and r_2 as follows:

$$\pi_i^{r_1} = \begin{cases} \underline{\pi}, & \text{if } \pi_i^r > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{ij}^{r_1} = \begin{cases} -\underline{\pi}, & \text{if } \pi_i^r > 0 \text{ and } \pi_j^r = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\pi_i^{r_2} = \pi_i^r - \pi_i^{r_1},$$
$$\alpha_{ij}^{r_2} = \alpha_{ij}^r - \alpha_{ij}^{r_1}.$$

Since there are at least two different values among the π variables in r, there is no real number λ such that $r_1 = \lambda r_2$. Therefore, r is not an extreme ray since it can be decomposed into two distinct rays. We have thus proven by contradiction that r can only be an extreme ray if all the positive π variables have the same value.

The condition on the α variables is straightforward. Indeed, let S be the set of nodes for which $\pi = 1$ and $\overline{S}$ the set of nodes for which $\pi = 0$. It is easy to see that only the α_{ij} variables with $i \in S$ and $j \in \overline{S}$ can have a value different from zero, and that their absolute value must be equal to the absolute value of π (otherwise this ray can be decomposed into two rays, one of which being composed only by α_{ij}). □

The fact that each extreme ray is associated with a cutset inequality brings us to an interesting remark: since the Benders feasibility cuts are sufficient to ensure feasibility and since

the extreme rays in the single-commodity case are all associated with cutset inequalities, one can conclude that cutset inequalities are sufficient to ensure feasibility in the single-commodity problem. This is the classical Ford-Fulkerson result.

In order to show that the form of the extreme rays obtained for the single-commodity case is not a general form for the MCFND, we use the counter-example presented in Figure 1. This example has two commodities (which are visually represented by a duplication of the graph). The values inside the nodes represent the π_i^k variables and the values close to the arcs the α_{ij} variables. Notice that there are positive π_i^k variables with different values and still the ray is an extreme ray.

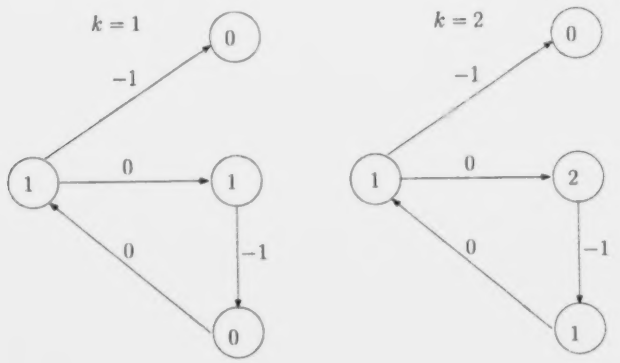

Figure 1: Extreme ray that does not follow the particular form of proposition 2

The example can be expanded to show that, in the general case, the π_i^k variables can assume an infinity of discrete values, with the discrete step being given by the value of the α_{ij} variables. For the general MCFND, the cutset inequalities are thus not sufficient to ensure feasibility, being only a subset of the feasibility constraints.

4 Benders feasibility cuts and metric inequalities

The space of feasible multicommodity flows was studied by Iri (1971) and Onaga and Kakusho (1971). The authors independently described the conditions for a set of inequalities to be necessary and sufficient to define this space. The result is sometimes referred to as the *Japanese theorem*, and is described in the following.

Let $\lambda = [\lambda_{ij}] \in \mathbb{R}_+^{|A|}$ be lengths associated with the arcs of the graph G and, considering these lengths, define $\theta(\lambda)$ as the weigthed sum (by the demands) of the lengths of the shortest

paths linking each commodity's origin and destination:

$$\theta(\lambda) = \sum_{k=1}^{K} d_k \ell_k(\lambda), \tag{21}$$

where $\ell_k(\lambda)$ is the length of the shortest path between $O(k)$ and $D(k)$ considering the graph where each arc (i,j) is given the length λ_{ij}. The Japanese theorem, which can easily be proven using strong duality theory, says that a graph with capacities $u_{ij} \geq 0$ will have feasible flows satisfying the demands if and only if:

$$\sum_{(i,j)\in A} u_{ij}\lambda_{ij} \geq \theta(\lambda), \quad \forall \lambda \in \mathbb{R}_+^{|A|}. \tag{22}$$

Inequalities (22) are known as *metric inequalities*. More on these inequalities can be found in the literature: Gondran and Minoux (1995) present the Japanese theorem within the context of an algorithm to determine the feasibility of a multicommodity flow, while Lomonosov (1985) presents an extensive article with general ideas based on the metric inequalities.

Let $y = [y_{ij}]$ be a tentative network and define $\overline{u}_{ij} = u_{ij}y_{ij}$. The Japanese theorem states that this tentative network is feasible if and only if:

$$\sum_{(i,j)\in A} \lambda_{ij}\overline{u}_{ij} \geq \theta(\lambda), \quad \forall \lambda \in \mathbb{R}_+^{|A|}. \tag{23}$$

We will now show that for each existing feasibility constraint of type (17), there exists a metric inequality that may make it redundant. The idea is simple: we show that the metric inequality has the same left-hand side, but a right-hand side that is at least as large as the one in the associated feasibility constraint.

Let $[\alpha, \pi]$ be an extreme ray of (11)-(12). On the one hand, the feasibility constraint generated by these dual variables is

$$\sum_{(i,j)\in A} -\alpha_{ij}\overline{u}_{ij} \geq \sum_{k\in K} d_k(\pi^k_{O(k)} - \pi^k_{D(k)}). \tag{24}$$

On the other hand, by making $\lambda_{ij} = -\alpha_{ij}$, one obtains the following metric inequality:

$$\sum_{(i,j)\in A} -\alpha_{ij}\overline{u}_{ij} \geq \theta(-\alpha_{ij}), \quad \forall \lambda \in \mathbb{R}_+^{A}. \tag{25}$$

Note that the left-hand side of metric inequality (25) is the same as the left-hand side of feasibility constraint (24). The following proposition shows that the right-hand side of the metric inequality is at least as large as the right-hand side of the feasibility cut.

Proposition 3 *For any extreme ray of (11)-(12), the values of the variables are such that* $\sum_{k \in K} d_k(\pi^k_{O(k)} - \pi^k_{D(k)}) \leq \theta(-\alpha_{ij})$.

Proof: We know that if $[\alpha, \pi]$ is a extreme ray, than the following inequalities are valid:

$$\pi^k_i - \pi^k_j + \alpha_{ij} \leq 0 \qquad \forall (i,j) \in A. \tag{26}$$

Consider a commodity k and write equation (26) for each arc in the shortest path between $O(k)$ and $D(k)$ (considering weights $-\alpha_{ij}$ for each arc). By multiplying each term by d_k, one obtains

$$d_k \pi^k_i - d_k \pi^k_j + d_k \alpha_{ij} \leq 0$$

and, by adding up all equations associated with arcs in a shortest path P^k,

$$d_k(\pi^k_{O(k)} - \pi^k_{D(k)}) + \sum_{(i,j) \in P^k} d_k \alpha_{ij} \leq 0. \tag{27}$$

By adding up equations (27) for all commodities, one finally obtains

$$\sum_{k \in K} d_k(\pi^k_{O(k)} - \pi^k_{D(k)}) \leq \sum_{k \in K} d_k \sum_{(i,j) \in P^k} (-\alpha_{ij})$$

or, equivalently, the desired result:

$$\sum_{k \in K} d_k(\pi^k_{O(k)} - \pi^k_{D(k)}) \leq \theta(-\alpha_{ij}).$$

□

Proposition 3 suggests a simple procedure to strengthen Benders feasibility cuts. For given values of the dual variables α_{ij}, the feasibility constraints can be lifted by solving $|K|$ shortest-path problems in a graph where the weight of each arc (i,j) is given by $-\alpha_{ij}$. The shortest path for commodity $k \in K$ is defined between $O(k)$ and $D(k)$.

This lifting procedure can be applied to the cutset inequalities, since they are a particular case of Benders feasibility constraints. The example below gives an interpretation of this result.

Example 1: Consider the network depicted in Figure 2, with four nodes (1, 2, 3 and 4) and 4 arcs (m, n, o and p). There are two commodities, with $[O(1), D(1), d_1] = [4, 2, d_1]$ and $[O(2), D(2), d_2] = [3, 2, d_2]$.

Consider the cutset given by the demand originated at $S = \{1, 3\}$ and destined to $\bar{S} = \{2, 4\}$. The dual variables that yield this cutset inequality are shown in Figure 3 and result in the feasibility cut $u_m y_m + u_p y_p \geq d_2$.

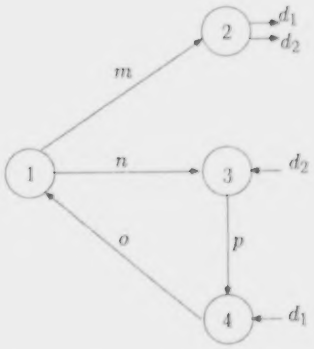

Figure 2: Network for example 1 - 4 nodes, 4 arcs and 2 commodities.

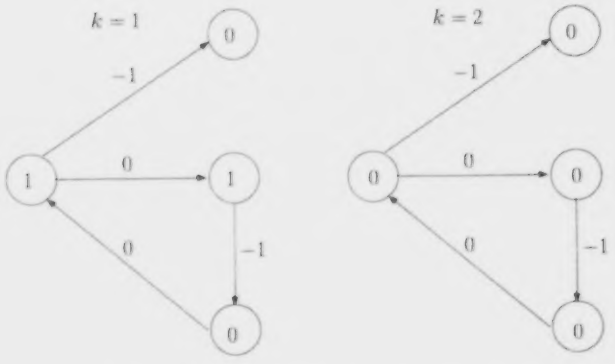

Figure 3: Dual variables that yield a feasibility cut equivalent to the cutset inequality for Example 1

Note, however, that if we consider $\lambda = -\alpha$, the value of the shortest path between $O(2)$ and $D(2)$ is equal to 2. Moreover, even though neither $O(1)$ nor $D(1)$ are in set S, the shortest path between $O(1)$ and $D(1)$ is equal to 1. Therefore, the lifting procedure yields the valid inequality $u_m y_m + u_p y_p \geq d_1 + 2d_2$, which clearly dominates the original cut.

The intuition behind this lifting procedure is very simple. Observe that in Example 1 the only path between $O(2)$ ($\in S$) and $D(2)$ ($\notin S$) must use more than one arc from the cutset (in this case, arcs p and m), because it needs to reenter the set S. Therefore, $\ell_k(-\alpha)$ actually represents the number of times *an arc belonging to the cutset* will have to be used in order for the commodity to leave the origin and reach the destination. The pure cutset inequality does not take this information into account (as if a single use of the cutset arcs would always be enough) and therefore presents a worse estimation of the needed capacity. The lifting procedure

also considers commodities for which neither the origin nor the destination is in the set S, but still need to use some capacity of the cutset, as is the case for commodity 1.

It is interesting to note that in this example, the same strengthened cut can be obtained as the sum of other cutset inequalities: the cut is obtained by adding up the cuts associated with the cutset $S = \{3\}, \overline{S} = \{1,2,4\}$ (which yields the inequality $u_p y_p \geq d_2$) and the cutset $S = \{1,3,4\}, \overline{S} = \{2\}$ (which yields the inequality $u_p y_p + u_m y_m \geq d_1 + d_2$). There are cases, however, in which one cannot obtain the strengthened inequality by adding up cutset inequalities, as shown in the next example.

Example 2: Consider the network of Example 1 with the addition of an arc q from node 3 to node 2. Again, we consider the cutset defined by $S = \{1,3\}$ and $\overline{S} = \{2,4\}$. The dual variables that yield this cutset inequality are shown in Figure 4 and result in the inequality $u_m y_m + u_p y_p + u_q y_q \geq d_2$. The lifted cutset inequality is $u_m y_m + u_p y_p + u_q y_q \geq d_1 + d_2$, which cannot be obtained as the sum of other cutset inequalities. Note, however, that it is dominated by the cutset inequality associated to $S = \{1,3,4\}$, which is $u_m y_m + u_q y_q >= d_1 + d_2$.

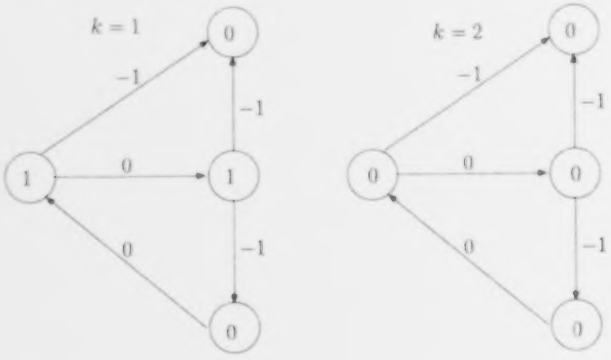

Figure 4: Dual variables that yield a feasibility cut equivalent to the cutset inequality for Example 2

The last two examples show that strenghtened cutset inequalities can sometimes be dominated by other cutset inequalities. In spite of this fact, the simple procedure we propose to strenghten cutset inequalities is interesting, because solution algorithms usually make use of heuristic methods to generate cutset inequalities. Hence, they do not guarantee that the best cutsets are generated. The proposed lifting procedure can be useful in that context, as our computational experiments, reported in Section 5.1, will confirm.

A final remark is that one could certainly try to generate metric inequalities directly instead of generating Benders feasibility cuts and strengthening them to metric inequalities. Indeed,

Gabrel, Knippel and Minoux (1999) have proposed a Benders decomposition algorithm for a multicommodity capacitated network design problem similar to the MCFND, where a heuristic method is used to generate violated metric inequalities. The disadvantage of this approach is that it is harder to obtain violated metric inequalities than to obtain violated feasibility constraints, in which case the solution of a linear problem is sufficient. The idea of our strengthening procedure is to take advantage of the easiness of obtaining feasibility constraints while benefitting from the strength of the metric inequalities.

5 Computational results

We tested the lifting procedure developed in the last section both for the case of the general feasibility constraints and for the particular case of the cutset inequalities. These tests were based on three sets of instances widely used in the literature. These instances contain from 10 to 100 nodes, 35 to 687 arcs and 10 to 400 commodities. They were described in detail by Ghamlouche, Crainic and Gendreau (2004).

5.1 Cutset inequalities

In the special case of the cutset inequalities, we have considered for each instance all sets S, with $|S| \leq \Psi$, Ψ being an integer which depends on the number of nodes in the graph (see Tables 1 and 2). All the associated cutsets have been generated and tested. In Table 1 the results are shown for the first group of instances (group R), while Table 2 shows the results obtained for the third group (group C+). For the second group (group C), considering sets with $|S| \leq 4$, no cutset inequality has been lifted. In the tables, for each instance, we show the following information:

- $(|N|, |A|, |K|)$ - Number of nodes, arcs and commodities in the instance;
- Ψ - maximum cardinality of the tested cutsets;
- Nb. Cutsets - number of cutsets tested;
- Nb. Lifted - number of cutsets that were strengthened by the procedure;
- Max Lift - the maximum lift obtained;
- Mean Lift - the average lift considering all the strengthened inequalities;

where the lift is defined as $\left[\theta(-\alpha_{ij}) - \sum_{k \in K} d_k \pi^k_{O(k)} - d_k \pi^k_{D(k)})\right] / \theta(-\alpha_{ij})$.

Table 1: Effect of the lifting procedure on cutsets (group R)

Inst.	$(\|N\|,\|A\|,\|K\|)$	Ψ	Nb. Cutsets	Nb. Lifted	Max Lift (%)	Mean Lift (%)
R01.1	(10,35,10)	5	637	32	42.11	32.35
R02.1	(10,35,25)	5	637	76	34.97	19.41
R03.1	(10,35,50)	5	637	63	24.02	10.51
R04.1	(10,60,10)	5	637	0	-	-
R05.1	(10,60,25)	5	637	0	-	-
R06.1	(10,60,50)	5	637	0	-	-
R07.1	(10,82,80)	5	637	0	-	-
R08.1	(10,83,25)	5	637	0	-	-
R09.1	(10,83,50)	5	637	0	-	-
R10.1	(20,120,40)	5	6195	2	14.70	14.18
R11.1	(20,120,100)	5	6195	48	12.26	4.28
R12.1	(20,120,200)	5	6195	67	4.98	2.92
R13.1	(20,220,40)	5	6195	0	-	-
R14.1	(20,220,100)	5	6195	0	-	-
R15.1	(20,220,200)	5	6195	0	-	-
R16.1	(20,314,40)	5	6195	0	-	-
R17.1	(20,318,100)	5	6195	0	-	-
R18.1	(20,315,200)	5	6195	0	-	-

Table 2: Effect of the lifting procedure on cutsets (group C+)

Inst.	$(\|N\|,\|A\|,\|K\|)$	Ψ	Nb. Cutsets	Nb. Lifted	Max Lift (%)	Mean Lift (%)
c25_100_10_F_L_5	(25,100,10)	4	15,275	22	100.00	45.71
c25_100_10_F_T_5	(25,100,10)	4	15,275	41	100.00	88.09
c25_100_10_V_L_5	(25,100,10)	4	15,275	160	100.00	65.77
c25_100_30_F_L_5	(25,100,30)	4	15,275	143	50.00	18.45
c25_100_30_F_T_5	(25,100,30)	4	15,275	868	100.00	25.07
c25_100_30_V_T_5	(25,100,30)	4	15,275	785	100.00	29.80
c100_400_10_F_L_10	(100,400,10)	3	166,750	3	100.00	84.84
c100_400_10_F_T_10	(100,400,10)	3	166,750	6	100.00	60.73
c100_400_10_V_L_10	(100,400,10)	3	166,750	98	100.00	56.00
c100_400_30_F_L_10	(100,400,30)	3	166,750	595	100.00	63.47
c100_400_30_F_T_10	(100,400,30)	3	166,750	304	100.00	75.91
c100_400_30_V_T_10	(100,400,30)	3	166,750	592	100.00	63.18

The results presented in Tables 1 and 2 indicate that the lifting procedure is only effective for sparse graphs. This result was expected, since the higher the arc density, the smaller the probability of the occurrence of the situation illustrated in Example 1. Indeed, consider a subset S of the nodes of a complete graph, and its complement $\overline{S}$. Starting in S, it is always possible to reach any node in $\overline{S}$ by using a single arc of the cutset and, therefore, the lifting procedure is useless.

5.2 Feasibility constraints

We have then studied the effect of the lifting procedure in the more general case of the feasibility constraints. For this purpose, we have developed a standard Benders decomposition algorithm which obtains values for the dual variables $[\alpha, \pi]$ by means of solving a linear subproblem. These variables are used in order to compare the efficiency of the lifting.

The Benders algorithm first relaxes all feasibility and optimality constraints in the master problem (16)-(19). At each iteration, the relaxed master problem provides a lower bound on the optimal solution value of the original problem and a solution $\overline{y}$. These variables define a tentative network, and are used in the dual subproblem (10)-(12). The dual subproblem is either unbounded, in which case the unbounded direction can be used to generate a violated constraint of type (17) (a feasibility cut), or bounded. In the latter case, the conjunction of the master problem and subproblem solutions is a feasible solution to the original problem (and provide an upper bound) and the extreme point can be used to generate a constraint of type (18) (an optimality cut). The process iterates until the values of the lower and upper bounds coincide.

We have analyzed the number of iterations and the time it takes for the Benders algorithm to find a first feasible solution. We compare these two figures with and without the lifting procedure. The number of iterations performed before a first feasible solution is found gives a measure of how much the lifting procedure can strengthen the cuts. The comparison of the computation time between the two cases is useful to determine if the time spent in the lifting procedure is compensated by the gain in the strength of the cuts. In Tables 3-7, the following results are shown:

- With or Without Lifting - indicates the use (or not) of the strengthening procedure:

  Nb. Iter - number of iterations needed to find a first feasible solution;

  Time (s) - time in seconds needed to find a first feasible solution;

- $\neq$ (%) - percentage difference between the equivalent columns with and without lifting;

- Lifting info - information associated with the strengthening procedure:

Nb. lifted - number of iterations where the cut was strengthened;

Max Lift - maximum lift obtained (%);

Mean Lift - mean lift obtained (%).

Table 3: Results for feasibility constraints (group R 10 nodes)

	Without Lifting		With Lifting		$\neq$ (%)		Lifting Info.		
Instance	Nb. Iter	Time (s)	Nb. Iter	Time (s)	Nb. Iter	Time	Nb. Lifted	Max Lift	Mean Lift
R01.1	45.50	0.10	37.67	0.09	-17.15	-9.14	11.00	28.81	16.06
R02.1	54.83	0.20	41.33	0.17	-24.90	-18.78	19.67	47.41	19.46
R03.1	58.50	0.39	44.50	0.31	-23.46	-20.49	29.00	48.79	19.41
R04.1	31.44	0.08	31.22	0.09	-0.60	12.53	0.33	5.64	16.91
R05.1	71.78	0.48	66.67	0.46	-6.61	-2.61	17.44	13.87	4.94
R06.1	94.56	1.49	76.78	1.19	-14.45	-16.23	33.78	20.33	6.82
R07.1	38.22	0.14	37.11	0.15	-2.89	4.86	1.89	7.63	6.18
R08.1	79.44	0.81	81.89	0.87	3.70	6.64	11.67	9.86	4.03
R09.1	117.22	2.79	96.22	2.02	-16.28	-25.87	36.25	25.80	5.86
Mean					-10.10	-6.40	17.25	20.87	9.61

Table 4: Results for feasibility constraints (group R 20 nodes)

	Without Lifting		With Lifting		$\neq$ (%)		Lifting Info.		
Instance	Nb. Iter	Time (s)	Nb. Iter	Time (s)	Nb. Iter	Time	Nb. Lifted	Max Lift	Mean Lift
R10.1	265.44	10.86	225.89	9.01	-12.66	-13.38	112.56	32.11	8.99
R11.1	282.33	48.18	204.44	34.04	-24.18	-27.83	136.44	62.92	14.57
R12.1	209.67	57.20	141.67	36.32	-30.02	-30.98	116.11	57.36	16.22
R13.1	431.78	97.43	399.89	91.72	-8.12	-6.84	117.89	17.51	3.99
R14.1	760.56	483.62	597.78	393.95	-17.93	-17.69	339.44	37.95	5.03
R15.1	542.00	708.65	353.78	483.58	-27.13	-23.70	255.22	62.34	7.83
R16.1	389.44	159.52	351.22	144.46	-9.60	-4.34	80.56	17.57	3.23
R17.1	1202.00	2315.23	983.78	1958.08	-18.05	-15.04	519.67	35.36	4.44
R18.1	978.89	3413.53	654.00	2471.57	-28.03	-26.11	446.22	35.34	4.86
Mean					-19.53	-18.44	236.01	39.83	7.68

Tables 3 and 4 show the results for the first group of instances. We see from these tables that the use of the procedure not only reduces the number of iterations needed to find a feasible solution for most instances, but also the time needed, indicating that the time spent to solve the shortest path problems is more than compensated by the strengthening of the cuts.

Table 5: Results for feasibility constraints (group C)

	Without Lifting		With Lifting		≠ (%)		Lifting Info.		
Instance	Nb. Iter	Time (s)	Nb. Iter	Time (s)	Nb. Iter	Time	Nb. Lifted	Max Lift	Mean Lift
c33	532	111.48	478	103.62	-10.15	-7.59	140	20.13	3.88
c35	454	141.42	393	120.56	-13.44	-17.30	88	16.54	3.40
c36	492	197.67	450	168.32	-8.54	-17.44	85	28.55	3.90
c37	2227	8041.07	1589	6298.42	-28.65	-27.67	1007	44.68	3.83
c38	2909	12738.5	1948	8906.8	-33.04	-43.02	1337	56.38	3.91
c39	2469	11205.6	1655	8391.61	-32.97	-33.53	999	47.60	4.06
c40	2653	12295.4	1750	9123.37	-34.04	-34.77	1105	58.41	3.17
c41	602	325.07	557	306.45	-7.48	-6.08	148	33.80	3.87
c42	588	331.64	553	282.48	-5.95	-17.40	68	15.35	3.17
c43	630	588.32	643	629.94	2.06	6.61	88	38.89	3.70
c44	911	893.24	793	658.8	-12.95	-35.59	107	13.71	3.08
c45	2959	18532.3	2179	14782	-26.36	-25.37	1367	46.06	2.65
c46	3024	18241.6	2206	14690.9	-27.05	-24.17	1300	32.49	2.60
c47	3222	21230	2193	15997.3	-31.94	-32.71	1294	45.43	2.61
c48	2772	15250.2	1991	12312.1	-28.17	-23.86	1127	52.36	2.71
Mean					-19.91	-22.66	684	36.69	3.37

Table 6: Results for feasibility constraints (group C - time limit exceeded)

	Without Lifting		With Lifting		≠ (%)	Lifting Info.		
Instance	Nb. Iter	LB	Nb. Iter	LB	LB	Nb. Lifted	Max Lift	Mean Lift
c49	849	3677.12	658	3658.39	-0.51	637	28.51	4.65
c50	1077	16381.4	766	16341.7	-0.24	736	32.41	4.65
c51	679	2895.79	549	2930.23	1.19	515	26.01	4.13
c52	760	17175.7	555	17320.2	0.84	529	26.76	4.62
c53	1061	10920.4	560	11977	9.68	553	78.01	11.56
c54	886	20131.3	472	21416.9	6.39	461	49.34	9.25
c55	769	11105.6	436	11956.4	7.66	423	78.92	8.30
c56	741	22470.3	421	24242.4	7.89	410	54.51	8.90
c57	388	2595.49	344	2676.2	3.11	317	24.44	4.59
c58	432	5140.63	347	5233.47	1.81	316	28.91	4.68
c59	501	2512.28	381	2538.63	1.05	352	27.53	3.80
c60	287	5271.1	236	5299.48	0.54	222	21.37	3.64
c61	690	8728.18	338	9245.4	5.93	314	63.75	13.24
c62	587	17684.2	317	19029.1	7.61	306	45.13	11.41
c63	628	8467.78	325	8988.69	6.15	310	49.36	9.33
c64	492	17317.9	264	18521.3	6.95	243	40.97	10.22
Mean					4.13	415.25	42.25	7.31

The results for the second group of instances are presented in Tables 5 and 6. These instances are harder than the first group. Indeed, for about half of these instances (those in Table 6), the Benders algorithm could not find a single feasible solution in the allowed computation time of 5 hours. Therefore Table 6 does not compare the difference between the total time to find a feasible solution, but the lower bound upon termination. Although it is a different measure, the

Table 7: Results for feasibility constraints (group C+)

	Without Lifting		With Lifting		$\neq$ (%)		Lifting Info.		
Instance	Nb. Iter	Time (s)	Nb. Iter	Time (s)	Nb. Iter	Time	Nb. Lifted	Max Lift	Mean Lift
c100_400_10_F_L_10	897	106.42	902	105.5	0.56	-0.86	56	21.88	9.84
c100_400_10_F_T_10	2170	491.34	2083	495.85	-4.01	0.92	59	33.49	11.62
c100_400_10_V_L_10	1361	312.08	1352	254.95	-0.66	-18.31	88	22.22	9.40
c100_400_30_F_L_10	6109	10871.3	5240	9678.83	-14.22	-10.97	1620	40.20	5.19
c100_400_30_F_T_10	3374	2361.29	3027	2085	-10.28	-11.70	1070	41.78	7.34
c25_100_10_F_L_5	171	1.06	158	1.09	-7.60	2.83	31	36.84	12.72
c25_100_10_F_T_5	133	0.71	127	0.73	-4.51	2.82	14	25.68	14.11
c25_100_10_V_L_5	115	0.64	95	0.54	-17.39	-15.63	25	25.81	17.07
c25_100_30_F_L_5	300	6.21	233	5.09	-22.33	-18.04	105	45.16	13.13
c25_100_30_F_T_5	226	3.7	164	2.86	-27.43	-22.70	99	69.80	22.32
c25_100_30_V_T_5	233	4.04	182	3.22	-21.89	-20.30	118	51.31	15.21
Mean					-11.80	-10.18	298.63	37.65	12.54

lower bounds are also relevant since they somehow indicate the strength of the cuts.

For the instances in Table 5, a mean reduction of 22% in the computation time needed to obtain the first feasible solution was obtained by using the lifting procedure. The results are consistent with those of the first group of instances and indicate that the harder the problem, the more effective the lifting procedure seems to be in reducing the total computation time.

An interesting result is obtained in Table 6. In this case, the time limit of 5 hours was not enough to obtain a single feasible solution. However, one observes that the lower bounds obtained by the Benders algorithm using the lifting procedure were slightly better (mean increase of 4.13%) than those obtained without the procedure. This indicates that the behavior is similar to the other two groups of instances. Finally, Table 7 shows the results for the last group of instances. Again, these results are similar to those for the other groups.

6 Conclusions

This article has studied the relations between three important families of valid inequalities for network design problems: the metric inequalities, the cutset inequalities and the Benders feasibility cuts. It has shown that cutset inequalities are a subset of Benders feasibility cuts, and that Benders feasibility cuts, including cutset inequalities, are not, in general, metric inequalities. By means of a simple procedure, which consists in computing $|K|$ shortest path problems, it has shown how to strengthen any Benders feasibility cut into the associated metric inequality. Computational results have shown that the procedure can be effective in practice. In some instances, the time needed to find a first feasible solution in a Benders decomposition algorithm was reduced by more than 30 %.

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